Problem Set Information

- Tutor: Nelson Lam (nelson.lam.lcy@gmail.com)
- References:
  - **Lecture Notes** of Dr. LAU & Dr. CHENG
  - Tutorial 2 Notes by Nelson Lam, Tutorial 1 Notes by James Chau
- Submission & Rewards
  - Complete at least 2 harder questions during tutorial
    Reward: Any drink from vending machine
- All the suggestions and feedback are welcome. Any report of typos is appreciated.

1 In Class Exercise

Exercise 1.1. Let $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

(a). Compute $\|\mathbf{u}\|, \|\mathbf{v}\|, \langle \mathbf{u}, \mathbf{v} \rangle$

(b). Compute the angle between $\mathbf{u}$ and $\mathbf{v}$

Exercise 1.2. Let $\mathbf{u} = 8\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

(a). Compute $\|\mathbf{u}\|, \|\mathbf{v}\|, \langle \mathbf{u}, \mathbf{v} \rangle$

(b). Compute $\text{Proj}_{\mathbf{v}}\mathbf{u}$

(c). Write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{v}$ and orthogonal to $\mathbf{v}$

Exercise 1.3 (Lecture Notes Chapter 1 Q2)**.** Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ with $\langle \mathbf{u}\, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle, \forall\, \mathbf{w} \in \mathbb{R}^3$

(a). Show that $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = 0$, for any choice of $\mathbf{w}$

(b). Use the fact that $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Rightarrow \mathbf{x} = \mathbf{0}$. Cleverly choose a vector $\mathbf{w}$ and substitute into (a). Conclude that $\mathbf{u} = \mathbf{v}$

Exercise 1.4 (2020 TDG Quiz 1 Q1 Modified)**.**
Let $\mathbf{u} = \frac{\sqrt{6}}{6}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} + \frac{\sqrt{6}}{3}\mathbf{k}, \mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}, \mathbf{w} = \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$ be 3 vectors in $\mathbb{R}^3$

(a). Show that they are mutually orthogonal

(b). Using (a), show that they constitutes an orthonormal basis for $\mathbb{R}^3$

(c). Express $\mathbf{r} = 2024\,\mathbf{i} + 8\,\mathbf{j} + 12\,\mathbf{k}$ as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Exercise 1.5. Let $\mathbf{u} = \frac{\sqrt{6}}{6}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} + \frac{\sqrt{6}}{3}\mathbf{k}, \mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

(a). Compute $\mathbf{u} \times \mathbf{v}$, is it a unit vector ?

(b). "If $\mathbf{w}$ is not given in Exercise 1.4, then there is insufficient data to finish Exercise 1.4", do you agree ?

Exercise 1.6 (2023 TDG Quiz 1 Q2).

(a). Evaluate the following determinants:

$$\det(R_z) = \det \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(R_y) = \det \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\det(R_x) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}$$

(b). Describe the geometric meaning of the three matrices R_x, R_y, R_z above.

(c). Describe the geometric meaning of the product $R_xR_yR_z$ of the three matrices above.

2 Warm up Question

Exercise 2.1. Let $\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{v} = \mathbf{i}-\mathbf{j}, \mathbf{w} = \mathbf{i}+\mathbf{j}-2\,\mathbf{k}$. Suppose $\mathbf{r} = \mathbf{i}+2\,\mathbf{j}+3\,\mathbf{k}$, find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\mathbf{r} = \alpha\,\mathbf{u}+\beta\,\mathbf{v}+\gamma\,\mathbf{w}$

Exercise 2.2. Let $\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{v} = \mathbf{i}-\mathbf{j}$ be two vectors in $\mathbb{R}^3$

(a). Is $\mathbf{u}$ orthogonal to $\mathbf{v}$?

(b). Compute $\mathbf{u} \times \mathbf{v}$

(c). Using (a), (b), construct an orthonormal basis for $\mathbb{R}^3$

Exercise 2.3 (2015 TDG Quiz 1 Q1)**.**
Let $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (3, 6, 9)$ be two vectors in $\mathbb{R}^3$

(a). Let $\mathbf{w} = \mathbf{v} - \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|^2}\,\mathbf{u}$, compute $\mathbf{w}$

(b). Show that $\mathbf{u}, \mathbf{w}$ are orthogonal

Exercise 2.4. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthogonal subset of $\mathbb{R}^m$. Suppose:

$$\mathbf{v} = \alpha_1\,\mathbf{v}_1 + \cdots + \alpha_k\,\mathbf{v}_k$$

$$\mathbf{w} = \beta_1\,\mathbf{v}_1 + \cdots + \beta_k\,\mathbf{v}_k$$

Show that for $\alpha_i, \beta_i \in \mathbb{R}, i = 1, 2, \ldots, k$, we have:

$$\langle \mathbf{v}, \mathbf{w}\rangle = \alpha_1\beta_1\|\mathbf{v}_1\|^2 + \cdots + \alpha_k\beta_k\|\mathbf{v}_k\|^2$$

Exercise 2.5 (2001 HKALE Pure Math Paper 1 Q7)**.**
A 2×2 matrix M is the matrix representation of a transformation T in $\mathbb{R}^2$. It is known that T transforms $(1, 0)$ and $(0, 1)$ to $(1, 1)$ and $(-1, 1)$ respectively.

(a). Find M

(b). Find $\lambda > 0$ and $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$ and M can be decomposed as:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Hence describe the geometric meaning of T

Exercise 2.6. For the following vector-valued functions, compute their derivatives:

(a). $\mathbf{r}(t) = (t^2 + 1, 2t, 9 - t)$

(b). $\mathbf{r}(t) = 3t[3t\,\mathbf{i} - 9t^2\,\mathbf{j} + (\cos t)\,\mathbf{k}\,]$

(c). $\mathbf{r}(t) = [(\sin t)\,\mathbf{i} - \cos(t)\,\mathbf{j}\,] \times [4t^7\,\mathbf{i} - 3t^2\,\mathbf{j} + (t^3 - t^2)\,\mathbf{k}\,]$

3 Standard Question

3.1 Orthogonalization & Orthogonal Basis

Exercise 3.1 (HKALE 1993 Pure Math Paper 1 Q8).
Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be linearly independent vectors in $\mathbb{R}^3$, show that:

(a). Suppose $\mathbf{s} \in \mathbb{R}^3$ with $\langle \mathbf{s}, \mathbf{u} \rangle = \langle \mathbf{s}, \mathbf{v} \rangle = \langle \mathbf{s}, \mathbf{w} \rangle = 0$, then $\mathbf{s} = \mathbf{0}$

(b). Suppose $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = 0$, prove that $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle = 0$

Deduce that: $\forall \mathbf{r} \in \mathbb{R}^3, \mathbf{r} = \dfrac{\langle \mathbf{r}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} + \dfrac{\langle \mathbf{r}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} + \dfrac{\langle \mathbf{r}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle} \mathbf{w}$

Exercise 3.2 (2003 HKALE Pure Math Paper 1 Q9).
Consider the vectors $\mathbf{a} = (p, q, 0), \mathbf{b} = (q, -p, 0), \mathbf{c} = (0, 0, r)$, where $p, q, r \in \mathbb{R} \backslash \{0\}$

(a). Prove that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent

(b). Let $\mathbf{d} \in \mathbb{R}^3$, prove that $\mathbf{d} = \dfrac{\langle \mathbf{d}, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a} + \dfrac{\langle \mathbf{d}, \mathbf{b} \rangle}{\langle \mathbf{b}, \mathbf{b} \rangle} \mathbf{b} + \dfrac{\langle \mathbf{d}, \mathbf{c} \rangle}{\langle \mathbf{c}, \mathbf{c} \rangle} \mathbf{c}$

(c). Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ are linearly independent, define $\mathbf{u} = \mathbf{x}$ and $\mathbf{v} = \mathbf{y} - \dfrac{\langle \mathbf{y}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$

Prove that $\mathbf{v}$ is a non-zero vector

(d). Define $\mathbf{w} = \mathbf{z} - \dfrac{\langle \mathbf{z}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} - \dfrac{\langle \mathbf{z}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$

- Prove that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are orthogonal
- Describe the geometric relationship between $\mathbf{w}$ and the plane containing the vectors $\mathbf{x}$ and $\mathbf{y}$

3.2 Orthogonal Matrix & Isometry

Exercise 3.3 (2023 TDG Quiz 1 Q4). Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

(a). Show that $\langle \mathbf{u}, \mathbf{v} \rangle = \dfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2)$

(b). Let A be a 3×3 matrix such that for any $\mathbf{v} \in \mathbb{R}^3$, $\|A\mathbf{v}\| = \|\mathbf{v}\|$. Show that for any $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$,

$$\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

(c). Write down any 3×3 matrix A except $\pm I$, such that for all $\mathbf{v} \in \mathbb{R}^3$, $\|A\mathbf{v}\| = \|\mathbf{v}\|$.

3.3 Vector-Valued Function & Differentiation By Part

Exercise 3.4 (Lecture Notes Chapter 1 Exercise 6).
Let $\mathbf{v}(t)$ be a differentiable vector-valued function. Suppose $\|\mathbf{v}\|$ is a constant independent of t, prove that $\dfrac{d\mathbf{v}}{dt}$ is orthogonal to $\mathbf{v}$ at any t

4 Harder Question

Exercise 4.1 (Reflection in $\mathbb{R}^2$ and $\mathbb{R}^3$, Credit to Houston Tang)**.** Let $\mathbf{u} \in \mathbb{R}^n$ be a fixed vector, define Householder Transformation:

$$H_{\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^n \text{ by } H_{\mathbf{u}}(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u}$$

(a). Show that

- $H_{\mathbf{u}}$ is linear
- $H_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} \iff \langle \mathbf{u}, \mathbf{x} \rangle = 0$
- $H_{\mathbf{u}}(\mathbf{u}) = -\mathbf{u}$
- $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \langle H_{\mathbf{u}}(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, H_{\mathbf{u}}(\mathbf{y}) \rangle$
- $H_{\mathbf{u}}$ is distance-preserving
- $H_{\mathbf{u}} \circ H_{\mathbf{u}} = \mathrm{Id}$

(b). By considering $\mathbf{u} = -(\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}$ and $n = 2$, derive the matrix representation of $H_{\mathbf{u}}$. It should be something you familiar.

(c). Generalise your answer to higher dimensions

Exercise 4.2 (Bessel's Inequality)**.** Denote $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^m$ be any arbitrary vector, where $m \geq n$. Show that:

$$\|\mathbf{x}\| \geq \left\| \mathbf{x} - \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i \right\|$$

5 Challenging Question

Exercise 5.1 (Lecture Notes Chapter 2 Exercise 11 Modified)**.**
Let $\mathbf{r}(s)$ be a differentiable vector-valued function on $\mathbb{R}^2$ with $\|\mathbf{r}'(s)\|= 1$. Denote $\mathbf{T}(s) = \mathbf{r}'(s)$

(a). Show that $\langle \mathbf{T}'(s), \mathbf{T}(s)\rangle = 0$

(b). Denote $\kappa(s) \stackrel{\text{def}}{=} \|\mathbf{T}'(s)\|$, define $\mathbf{N}(s)$ by the relation: $\mathbf{T}'(s) = \kappa(s)\,\mathbf{N}(s)$
Compute $\|\mathbf{N}(s)\|$ and $\langle \mathbf{T}(s), \mathbf{N}(s)\rangle$, deduce that $\langle \mathbf{T}(s), \mathbf{N}'(s)\rangle = -\kappa(s)$

(c). Does $\{\mathbf{T}(s), \mathbf{N}(s)\}$ constitute a orthonormal basis for $\mathbb{R}^2$? Hence prove or disprove:

$$\mathbf{N}'(s) = -\kappa(s)\,\mathbf{T}(s)$$